GLOBAL DYNAMICS OF THE CHEMOSTAT WITH DIFFERENT REMOVAL RATES AND VARIABLE YIELDS

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Abstract. In this paper, we consider a competition model between n species in a chemostat including both monotone and non-monotone growth functions, distinct removal rates and variable yields. We show that only the species with the lowest break-even concentration survives, provided that additional technical conditions on the growth functions and yields are satisfied. We construct a Lyapunov function which reduces to the Lyapunov function used by S. B. Hsu [SIAM J. Appl. Math., 34 (1978), pp. 760-763] in the Monod case when the growth functions are of Michaelis-Menten type and the yields are constant. Various applications are given including linear, quadratic and cubic yields.

Key words. chemostat, competitive exclusion principle, Lyapunov function, global asymptotic stability, variable yield model

AMS subject classifications. 92A15, 92A17, 34C15, 34C35

1. Introduction. In this paper we study the global dynamics of the following model of the chemostat in which n populations of microorganisms compete for a single growth-limiting substrate:

$$\begin{aligned} S'(t) &= D[S^0 - S(t)] - \textstyle\sum_{i=1}^{n} f_i(S(t))x_i(t) \\ x_i'(t) &= [p_i(S(t)) - D_i]x_i(t), \qquad\qquad i = 1 \cdots n, \end{aligned} \tag{1.1}$$

where $S(0) \geq 0$ and $x_i(0) > 0$, $i = 1 \cdots n$ and S^0, D and D_i are positive constants. In these equations, $S(t)$ denotes the concentration of the substrate at time t; $x_i(t)$ denotes the concentration of the ith population of microorganisms at time t; $f_i(S)$ represents the uptake rate of substrate of the ith population; $p_i(S)$ represents the per-capita growth rate of the ith population and so the function $y_i(S)$, defined by $y_i(S) = \frac{p_i(S)}{f_i(S)}$ is the growth yield; S^0 and D denote, respectively, the concentration of substrate in the feed bottle and the flow rate of the chemostat; each D_i represents the removal rate of the ith population. For general background on model (1.1), in the constant yield case $y_i(S) = Y_i$, the reader is referred to the monograph of Smith and Waltman [19].

The global analysis of this model was considered by Hsu, Hubbell and Waltman [6], in the Monod case [15] when the growth functions are of Michaelis-Menten form,

$$p_i(S) = \frac{a_i S}{b_i + S}, \tag{1.2}$$

and the yields are constant $y_i(S) = Y_i$, and $D_i = D$ for $i = 1 \cdots n$. The authors showed that only the species with the lowest break-even concentration survives. Thus the competitive exclusion principle (CEP) holds: only one species survives, namely the species which makes optimal use of the resources. Hsu [5] applied a Lyapunov

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argument to give a simple and elegant proof of the result in [6] for the case of different removal rates D_i. The Lyapunov function V_H used by Hsu is

$$V_H = \int_{\lambda_1}^{S} \frac{\sigma - \lambda_1}{\sigma} d\sigma + c_1 \int_{x_1^*}^{x_1} \frac{\xi - x_1^*}{\xi} d\xi + \sum_{i=2}^{n} c_i x_i, \tag{1.3}$$

where $c_i = \frac{1}{Y_i} \frac{a_i}{a_i - D_i}$, $i = 1 \cdots n$, $x_1^* = DY_1 \frac{S^0 - \lambda_1}{D_1}$ and $\lambda_1 = \frac{b_1 D_1}{a_1 - D_1}$ is the lowest break-even concentration of the species.

Wolkowicz and Lu [21] extended the results of [5] by allowing more general growth functions. These authors used the Lyapunov function

$$V_{WL} = \frac{S^0 - \lambda_1}{D_1} \int_{\lambda_1}^{S} \frac{p_1(\sigma) - D_1}{S^0 - \sigma} d\sigma + \frac{1}{Y_1} \int_{x_1^*}^{x_1} \frac{\xi - x_1^*}{\xi} d\xi + \sum_{i=2}^{n} \frac{\alpha_i}{Y_i} x_i, \tag{1.4}$$

where α_i, $i = 2 \cdots n$ are positive constants to be determined. They identified a large class of growth functions, including many prototypes of growth functions often found in the literature, where the constant α_i in (1.4) can always be found. Despite the fact the α_i cannot be found for all growth functions, the work of Wolkowicz and Lu [21] represents a major step in the extension of the result of Hsu [5] to general growth functions.

The CEP has also been proved under a variety of hypotheses by Armstrong and McGehee [2], Butler and Wolkowicz [3], Wolkowicz and Xia [22] and Li [9]. The hypotheses used in [2, 3, 6, 5, 9, 21, 22] are summarized in Table 1 of [8]. However, the problem is not yet completely solved: the CEP holds for a large class of growth functions but an important open question remains: *is the CEP true assuming only that the f_i are monotone with no restriction on the D_i ?* This major open problem remains unresolved, see in particular [8, 20]. For other studies and complements on the use of Lyapunov techniques in the chemostat, see [4, 10, 12, 13, 14].

The variable yield case was considered, for $n = 1, 2$ by Pilyugin and Waltman [16], with a particular interest to linear and quadratic yields, and by Huang, Zhu and Chang [7]. The model (1.1), with variable yields, was considered by Arino, Pilyugin and Wolkowicz [1]. For biological motivations concerning the dependence of the yields on the substrate, see [1, 16] and the references therein.

Notice that, in the case when the growth functions are of Michaelis-Menten form (1.2), the Lyapunov function (1.4) does not reduce to the Lyapunov function (1.3). Our aim in this paper is to extend the Lyapunov function (1.3) of Hsu [5] to the chemostat with a more general class of growth functions and variable yields. Our Lyapunov function is given by

$$V = \int_{\lambda_1}^{S} \frac{p_1(\sigma) - D_1}{f_1(\sigma)} d\sigma + \int_{x_1^*}^{x_1} \frac{\xi - x_1^*}{\xi} d\xi + \sum_{i=2}^{n} \alpha_i x_i \tag{1.5}$$

where α_i, $i = 2 \cdots n$ are positive constants to be determined. This Lyapunov function is just a multiple of the Lyapunov function (1.3) that Hsu used in [5] in the Monod case, see Section 3.1. It is also a multiple of the one used in [22] Page 1039 or [20] Section 3.3, in the case of one species, general growth function and constant yield, see Section 3.2.

The paper is organized as follows. In section 2 we prove our main result (see Theorem 2.2) and we compare it with the main result in [21] (see Theorem 2.3), where the yields are assumed to be constant. It should be noticed that, in the case

when the yields are constant, our result follows from the result in [21]. Actually, both theorems 2.2 and 2.3 are corollaries of a more general result, which is valid in the case when the yields are variable [18]. In Section 3.1 we consider the Monod model with constant yields. In Section 3.2 we consider the one species case and we show that our Lyapunov function can be used to obtain the same result as in [1]. In Section 3.3 we show that for the Monod model with constant yields replaced by either linear or quadratic functions of S, under certain additional technical assumptions, the CEP still holds (see Corollary 3.1). In Section 3.4 we consider the model of Pilyugin and Waltman [16] which was used to demonstrate that a periodic orbit was possible in the case of variable yield model. In this model, with two species, where one yield is constant and the other is cubic in S, we show that our Lyapunov function can be used to prove that for some values of the parameters the CEP holds (see Corollary 3.2). In Section 3.5 we identify a class of growth functions, including Lotka-Volterra and Michaelis-Menten growth functions where our Lyapunov function works. Concluding remarks are given in Section 4.

2. Global asymptotic stability. We make the following assumptions on the functions p_i and f_i:

- $p_i, f_i : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ are continuous,
- $p_i(0) = f_i(0) = 0$ and for all $S > 0$, $p_i(S) > 0$ and $f_i(S) > 0$.

Following Butler and Wolkowicz [3], we make the following assumptions on the form of the growth functions p_i: there exist positive extended real numbers λ_i and μ_i with $\lambda_i \leq \mu_i \leq +\infty$ such that

$$p_i(S) < D_i \text{ if } S \notin [\lambda_i, \mu_i], \text{ and } p_i(S) > D_i \text{ if } S \in]\lambda_i, \mu_i[.$$

Hence there are at most two values, $S = \lambda_i$ and $S = \mu_i$, called the break-even concentrations, satisfying the equation $p_i(S) = D_i$. We adopt the convention $\mu_i = \infty$ if this equation has only one solution and $\lambda_i = \infty$ if it has no solution.

It is known (see Theorem 4.1 [1]) that the non-negative cone is invariant under the flow of (1.1) and all solutions are defined and remain bounded for all $t \geq 0$. System (1.1) can have many equilibria: the washout equilibrium $E_0 = (S^0, 0, \cdots, 0)$, which is locally exponentially stable if and only if for all $i = 1 \cdots n$, $S^0 \notin [\lambda_i, \mu_i]$ and the equilibria E_i^* and E_i^{**} where all components of E_i^* and E_i^{**} vanish except for the first and the $(i+1)$th, which are

$$S = \lambda_i,\ x_i = x_i^* = F_i(\lambda_i), \text{ for } E_i^* \text{ and } S = \mu_i,\ x_i = x_i^{**} = F_i(\mu_i), \text{ for } E_i^{**}$$

respectively, where $F_i(S) = D\dfrac{S^0 - S}{f_i(S)}$.

The equilibrium E_i^* lies in the non-negative cone if and only if $\lambda_i \leq S^0$. If $\lambda_i < \lambda_j$ for all $i \neq j$ and $F_i'(\lambda_i) < 0$ then it is locally exponentially stable. It coalesces with E_0 when $\lambda_i = S^0$. The equilibrium E_i^{**} lies in the non-negative cone if and only if $\mu_i \leq S^0$ and is locally exponentially unstable if it exists. Its coalesces with E_0 when $\mu_i = S^0$. Besides these equilibria, the system (1.1) can have a continuous set of non-isolated equilibria in the non-generic cases where two or more of the break-even concentrations are equal. In what follows we assume, that $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$, and $\lambda_1 < S^0 < \mu_1$. Hence E_0 is locally exponentially unstable and the equilibrium $E_1^* = (\lambda_1, x_1^*, 0, \cdots, 0)$, where $x_1^* = F_1(\lambda_1) = D\frac{S^0 - \lambda_1}{f_1(\lambda_1)}$, lies in the non-negative cone. It is locally exponentially stable if and only if $F_1'(\lambda_1) < 0$. We consider the global asymptotic stability of E_1^*.

Before presenting the results, we need the following lemma,

LEMMA 2.1. *The solutions $S(t)$, $x_i(t)$, $i = 1 \cdots n$ of (1.1) with positive initial conditions are positive and bounded, and if $\lambda_i < S^0 < \mu_i$ for some $i = 1 \cdots n$, then $S(t) < S^0$ for all sufficiently large t.*

Proof. The proof is similar to the proof of Lemma 2.1 in [21] obtained for the model (1.1) in the case where the yields are constant. □

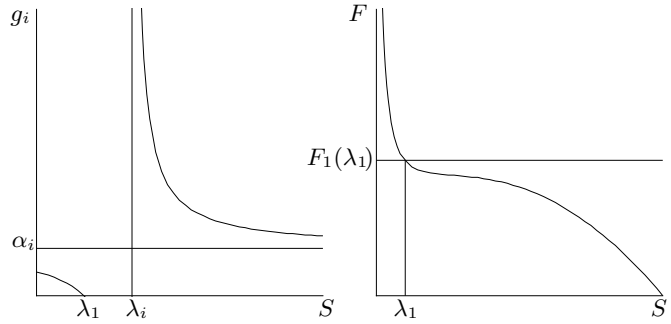


FIG. 2.1. *Graphical depictions of the hypotheses* 2 *and* 3 *in Theorem 2.2. On the left, hypothesis* 2. *On the right, hypothesis* 3.

We have the following result.

THEOREM 2.2. *Assume that*

1. $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$, *and* $\lambda_1 < S^0 < \mu_1$.
2. *There exist constants* $\alpha_i > 0$ *for each* $i \geq 2$ *satisfying* $\lambda_i < S^0$, *such that*

$$\max_{0<S<\lambda_1} g_i(S) \leq \alpha_i \leq \min_{\lambda_i<S<\rho_i} g_i(S), \tag{2.1}$$

   where $g_i(S) = \frac{f_i(S)}{f_1(S)} \frac{p_1(S)-D_1}{p_i(S)-D_i}$ *and* $\rho_i = \min(\mu_i, S^0)$.
3. *The function* $F_1(S) = D\frac{S^0-S}{f_1(S)}$ *satisfies* $F_1(S) > F_1(\lambda_1)$ *if* $S \in]0, \lambda_1[$, *and* $F_1(S) < F_1(\lambda_1)$ *if* $S \in]\lambda_1, S^0[$.

Then the equilibrium E_1^* is globally asymptotically stable for system (1.1) with respect to the interior of the positive cone.

Proof. From Lemma 2.1 it follows that there is no loss of generality in restricting our attention to $0 \leq S < S^0$. Consider the function $V = V(S, x_1, \cdots, x_n)$ given by (1.5), where α_i are the positive constants satisfying (2.1). The function V is continuously differentiable in the positive cone and positive except at the point E_1^*, where it is equal to 0. The derivative of V along the trajectories of (1.1) is given by

$$V' = [p_1(S) - D_1]\,[F_1(S) - x_1^*] + \sum_{i=2}^{n} x_i \theta_i(S),$$

where $\theta_i(S) = [p_i(S) - D_i]\,[\alpha_i - g_i(S)]$. First, note that, using hypotheses *1* and *3*, the first term of the above sum is always non-positive for $0 < S < S^0$ and equals 0 for $S \in]0, S^0[$ if and only if $S = \lambda_1$. If $S \in [\lambda_1, \lambda_i]$ then $p_i(S) < D_i$ and $p_1(S) > D_1$ so that $g_i(S) < 0 < \alpha_i$ for any choice of $\alpha_i > 0$. Similarly if $\mu_i < S^0$ and $S \in [\mu_i, S^0]$ then $p_i(S) < D_i$ and $p_1(S) > D_1$ so that $g_i(S) < 0 < \alpha_i$ for any choice of $\alpha_i > 0$. On the other hand, if $S \in [0, \lambda_1]$ then $p_i(S) < D_i$ and, using (2.1), $g_i(S) \leq \alpha_i$ so that $\theta_i(S) < 0$. Finally, if $S \in [\lambda_i, \rho_i]$ then $p_i(S) > D_i$ and $g_i(S) \geq \alpha_i$ so that $\theta_i(S) < 0$. Thus $\theta_i(S) < 0$ for every $S \in]0, S^0[$, provided that the numbers α_i satisfy

(2.1). Hence $V' \leq 0$ and $V' = 0$ if and only if $S = \lambda_1$ and $x_i = 0$ for $i = 2 \cdots n$. By the Krasovskii-LaSalle extension Theorem, the ω-limit set of the trajectory is E_1^*. □
In the case when the yields are constant, $y_i(S) = Y_i$, (1.1) takes the form

$$S' = D[S^0 - S] - \sum_{i=1}^{n} \frac{p_i(S)}{Y_i} x_i, \qquad x_i' = [p_i(S) - D_i]x_i, \quad i = 1 \cdots n. \tag{2.2}$$

Using the Lyapunov function (1.4), Wolkowicz and Lu (see Theorem 2.3 in [21]) proved the following result

THEOREM 2.3. *Assume that*

1. $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$, *and* $\lambda_1 < S^0 < \mu_1$.
2. *There exist constants* $\alpha_i^{WL} > 0$ *for each* $i \geq 2$ *satisfying* $\lambda_i < S^0$ *such that*

$$\max_{0<S<\lambda_1} g_i^{WL}(S) \leq \alpha_i^{WL} \leq \min_{\lambda_i<S<\rho_i} g_i^{WL}(S), \tag{2.3}$$

where $g_i^{WL}(S) = \frac{p_i(S)}{D_1} \frac{p_1(S)-D_1}{p_i(S)-D_i} \frac{S^0-\lambda_1}{S^0-S}$ *and* $\rho_i = \min(\mu_i, S^0)$.

Then the equilibrium E_1^* *is globally asymptotically stable for system (2.2) with respect to the interior of the positive cone.* It should be noticed that, in the case when the yields are constant, conditions (2.3) are consequences of conditions (2.1) in Theorem 2.2. Indeed, we have

$$g_i^{WL}(S) = \frac{(S^0 - \lambda_1)Y_i}{D_1} \frac{f_1(S)}{S^0 - S} g_i(S) = \frac{(S^0 - \lambda_1)Y_i D}{D_1} \frac{g_i(S)}{F_1(S)}.$$

Thus, hypotheses *2* and *3* of Theorem 2.2 imply hypothesis *2* of Theorem 2.3. Hence, in the case when the yields are constant, Theorem 2.2 follows from Theorem 2.3.

It is of interest to identify classes of growth functions where conditions (2.1) are satisfied, and hence Theorem 2.2 can be applied. We give below a result which will be used in the following section to verify easily that conditions (2.1) are satisfied. This proposition is similar to Corollary 2.4 in [21].

PROPOSITION 2.4. *Suppose that for each* $i \geq 2$, $w_i(S) = g_i(S)\frac{S-\lambda_i}{S-\lambda_1}$ *satisfies*

$$\max_{0<S<\lambda_1} w_i(S) \leq \min_{\lambda_i<S<\rho_i} w_i(S). \tag{2.4}$$

Then conditions (2.1) are satisfied.

Proof. The function $h_i(S) = \frac{S-\lambda_1}{S-\lambda_i}$ is decreasing on $[0, \lambda_1]$ and on $]\lambda_i, +\infty[$ and admits 1 as a horizontal asymptote. Thus (see Figure 3.1, right)

$$\max_{0<S<\lambda_1} h_i(S) = h_i(0) < 1 < h_i(\rho_i) = \min_{\lambda_i<S<\rho_i} h_i(S). \tag{2.5}$$

By (2.4) and (2.5) the functions $g_i(S) = w_i(S)h_i(S)$ satisfy (2.1). □

3. Applications. In this section we show how Theorem 2.2 can be fruitfully used to analyze the stability properties of systems whose yield functions depend on the variable S. We begin with the classical Monod case where the yields are constant and the growth functions are of Michaelis-Menten form.

3.1. The Monod case. Consider the particular case where the growth functions $p_i(S)$ are given by (1.2) and the yields are constant. System (1.1) takes the form

$$S' = D(S^0 - S) - \sum_{i=1}^{n} \frac{a_i S}{b_i + S} \frac{x_i}{Y_i}, \qquad x_i' = \left[\frac{a_i S}{b_i + S} - D_i\right] x_i, \quad i = 1 \cdots n. \tag{3.1}$$

We consider the case where, for all $i = 1 \cdots n$, $a_i > D_i$. In that case:

$$\lambda_i = \frac{b_i D_i}{a_i - D_i}, \quad \mu_i = \infty, \qquad g_i(S) = w_i \frac{S - \lambda_1}{S - \lambda_i}, \quad \text{where} \quad w_i = \frac{a_i Y_1 (a_1 - D_1)}{a_1 Y_i (a_i - D_i)}.$$

By Proposition 2.4, the conditions (2.1) are satisfied. Since

$$F_1(S) = Y_1 D(S^0 - S)\frac{b_1 + S}{a_1 S} \qquad \text{and} \qquad F_1'(S) = -Y_1 D \frac{S^2 + b_1 S^0}{a_1 S^2},$$

the first derivative of the function $F_1(S)$ is negative. Hence, hypothesis *3* in Theorem 2.2 is satisfied. The global stability of the equilibrium E_1^* of (3.1) follows from Theorem 2.2. This result was obtained by Hsu [5], using the Lyapunov function (1.3). Notice that, in this case, the Lyapunov function (1.5) is simply $V = \frac{a_1 - D_1}{a_1 Y_1} V_H$ where V_H is the Lyapunov function (1.3) used by Hsu [5].

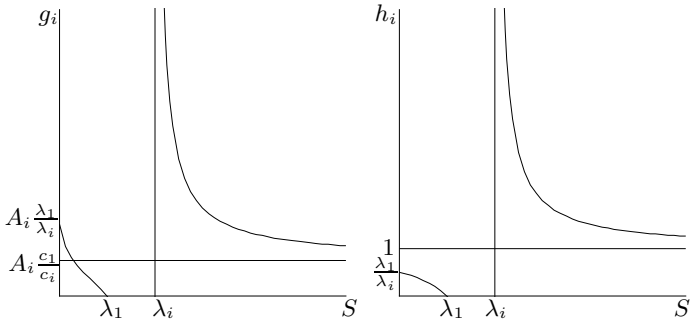


FIG. 3.1. *On the left, the graph of the function $g_i(S)$ for $p_i(S) = \frac{a_i S}{b_i + S}$ and $y_i(S) = Y_i(1 + c_i S)$: hypothesis* 2 *in Corollary 3.1 is not satisfied. On the right the graph of the function $h_i(S) = \frac{S - \lambda_1}{S - \lambda_i}$.*

3.2. One species. In the case $n = 1$, (1.1) takes the form

$$S' = D(S^0 - S) - x_1 f_1(S), \qquad x_1' = [p_1(S) - D_1]x_1 \tag{3.2}$$

If $\lambda_1 < S^0 < \mu_1$ and hypothesis *3* in Theorem 2.2 is satisfied then the equilibrium $E_1^* = (\lambda_1, x_1^*)$ of (3.2), where $x_1^* = D\frac{S^0 - \lambda_1}{f_1(\lambda_1)}$ is globally asymptotically stable with respect to the interior of the positive quadrant. This result follows from Theorem 2.2 since in the case where $n = 1$ the condition (2.1) is obviously satisfied. The global asymptotic stability of E_1^* was obtained previously by Arino, Pilyugin and Wolkowicz [1]. These authors used the following Lyapunov function

$$V_{APW} = \frac{S^0 - \lambda_1}{f_1(\lambda_1)} \int_{\lambda_1}^{S} \frac{p_1(\sigma) - D_1}{S^0 - \sigma} d\sigma + \int_{x_1^*}^{x_1} \frac{\xi - x_1^*}{\xi} d\xi.$$

They proved (see [1], Theorem 2.11) that if $1 - \frac{f_1(S)(S^0 - \lambda_1)}{f_1(\lambda_1)(S^0 - S)}$ has exactly one sign change for $S \in (0, S^0)$ then E_1^* is globally asymptotically stable. The condition on the change of sign is equivalent to hypothesis *3* in Theorem 2.2. Notice that the Lyapunov function we obtain is not proportional to the Lyapunov function V_{APW} considered in [1]. However, in the case when the yields is constant the global asymptotic stability of the equilibrium E_1^* of the system

$$S' = D(S^0 - S) - x_1 \frac{p_1(S)}{Y_1}, \qquad x_1' = [p_1(S) - D_1]x_1,$$

can be obtained, using the following Lyapunov function (see [22] page 1039 or [20], Section 3.3)

$$V_{WBL} = \int_{\lambda_1}^{S} \frac{p_1(\sigma) - D_1}{p_1(\sigma)} d\sigma + \frac{1}{Y_1} \int_{x_1^*}^{x_1} \frac{\xi - x_1^*}{\xi} d\xi.$$

In this case we simply have $V = Y_1 V_{WBL}$, where V is our Lyapunov function (1.5).

3.3. Michaelis-Menten growth functions and linear or quadratic yields. Consider the particular case of (1.1), where the growth functions $p_i(S)$ are given by (1.2), and the yields $y_i(S) = p_i(S)/f_i(S)$ are linear

$$y_i(S) = Y_i(1 + c_i S) \tag{3.3}$$

or quadratic

$$y_i(S) = Y_i(1 + c_i S^2). \tag{3.4}$$

where $Y_i > 0$ and $c_i \geq 0$. System (1.1) takes the form

$$S' = D(S^0 - S) - \sum_{i=1}^{n} \frac{a_i S}{b_i + S} \frac{x_i}{y_i(S)}, \qquad x_i' = \left[\frac{a_i S}{b_i + S} - D_i \right] x_i, \quad i = 1 \cdots n. \tag{3.5}$$

COROLLARY 3.1. *Consider system (3.5) where the yields are given by (3.3) or (3.4). Assume that*

1. $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ *and* $\lambda_1 < S^0$, *where* $\lambda_i = \frac{b_i D_i}{a_i - D_i}$.
2. *For each* $i \geq 2$ *satisfying* $\lambda_i < S^0$ *we have* $c_i \lambda_1 \leq c_1 \lambda_i$.
3. *The function* $F_1(S) = D \frac{S^0 - S}{a_1 S}(b_1 + S) y_1(S)$ *satisfies* $F_1(S) > F_1(\lambda_1)$ *if* $S \in]0, \lambda_1[$, *and* $F_1(S) < F_1(\lambda_1)$ *if* $S \in]\lambda_1, S^0[$.

Then the equilibrium E_1^* *is globally asymptotically stable for (3.5) with respect to the interior of the positive cone.*

Proof. For linear yields (3.3) we have

$$g_i(S) = w_i(S) \frac{S - \lambda_1}{S - \lambda_i}, \text{ where } w_i(S) = A_i \frac{1 + c_1 S}{1 + c_i S}, \quad A_i = \frac{a_i Y_1 (a_1 - D_1)}{a_1 Y_i (a_i - D_i)}.$$

Two cases can be distinguished. If $c_1 \geq c_i$ then the function $w_i(S)$ is non-decreasing over $[0, +\infty[$. Thus

$$\max_{0 < S < \lambda_1} w_i(S) \leq \min_{\lambda_i < S < \rho_i} w_i(S).$$

By Proposition 2.4 the functions $g_i(S) = w_i(S) h_i(S)$ satisfy (2.1). If $c_1 < c_i$ then from the expression

$$g_i'(S) = A_i \frac{c_1 - c_i}{(1 + c_i S)^2} \frac{S - \lambda_1}{S - \lambda_i} + A_i \frac{1 + c_1 S}{1 + c_i S} \frac{\lambda_1 - \lambda_i}{(S - \lambda_i)^2},$$

we deduce that $g_i'(S) < 0$ for all $0 \leq S \leq \lambda_1$ and $S > \lambda_i$. Hence (see Fig. 3.1, left)

$$\max_{0 < S < \lambda_1} g_i(S) = g_i(0) = A_i \frac{\lambda_1}{\lambda_i} \text{ and } \min_{\lambda_i < S < S^0} g_i(S) = g_i(S^0) > g_i(+\infty) = A_i \frac{c_1}{c_i}.$$

Under hypothesis *2* there exists α_i satisfying (2.1). The result follows by Theorem 2.2. For quadratic yields (3.4) we have

$$g_i(S) = w_i(S)\frac{S-\lambda_1}{S-\lambda_i}, \text{ where } w_i(S) = A_i\frac{1+c_1S^2}{1+c_iS^2}, \quad A_i = \frac{a_iY_1(a_1-D_1)}{a_1Y_i(a_i-D_i)}.$$

Thus

$$g_i'(S) = A_i\frac{2(c_1-c_i)S}{(1+c_iS^2)^2}\frac{S-\lambda_1}{S-\lambda_i} + A_i\frac{1+c_1S^2}{1+c_iS^2}\frac{\lambda_1-\lambda_i}{(S-\lambda_i)^2}.$$

Next, the proof is *mutatis mutandis* the same as the proof given above for the case of linear yields (3.3). □

This result contains as a particular case the result of Hsu [5] which corresponds to the case where the yields are constant. Indeed, for constant yields $c_i = 0$, so that hypotheses *2* and *3* in Corollary 3.1 are satisfied.

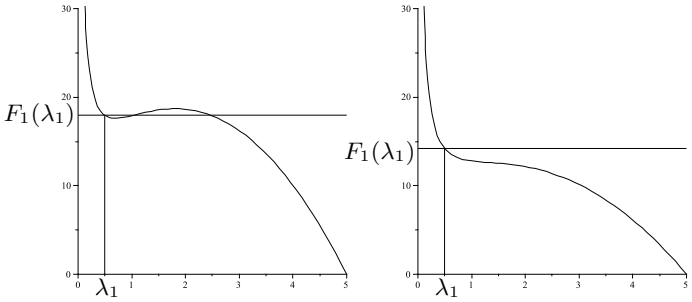


FIG. 3.2. *The graph of the function $F_1(S)$ for $p_1(S) = \frac{a_1S}{b_1+S}$ (where $a_1 = 2.1/1.1$, $b_1 = 0.5/1.1$ and $\lambda_1 = 0.5$) and $y_1(S) = Y_1(1+c_1S)$. On the left, the case $c_1 = 1.8$ for which hypothesis 3 in Corollary 3.1 is not satisfied. On the right, the case $c_1 = 1$ for which this hypothesis is satisfied.*

Remark. For linear or quadratic yields the function $F_1(S)$ is not monotone in general on the interval $]0, S^0[$, and it is not easy to give a condition on the parameters for which hypothesis *3* in Corollary 3.1 holds. However, in each example, the graphical depiction of this hypothesis is very simple as shown in Fig. 3.2.

3.4. Pilyugin-Waltman's example. This system was given in [16] as a model of the competition in the chemostat exhibiting limit cycles. The existence of the limit cycles is a consequence of the variable yield in the model. The model takes the form

$$\begin{aligned} S' &= 1 - S - \tfrac{2S}{0.7+S}\tfrac{x_1}{1+cS^3} - \tfrac{m_2S}{6.5+S}\tfrac{x_2}{120} \\ x_1' &= [\tfrac{2S}{0.7+S} - 1]x_1 \\ x_2' &= [\tfrac{m_2S}{6.5+S} - 1]x_2. \end{aligned} \tag{3.6}$$

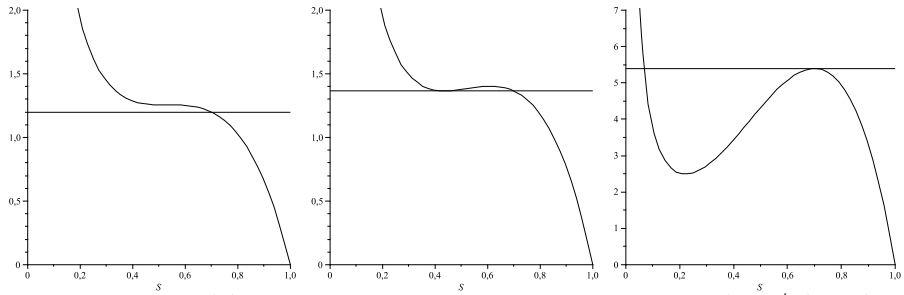
In their study Pilyugin and Waltman [16] fixed $c = 50$ and considered m_2 as a bifurcation parameter. They showed that for $m_2 \geq 9.85$ the system exhibits sustained oscillations. In this section we fix $m_2 = 10$ and we consider $c \geq 0$ as a bifurcation parameter. In this case we have

$$\lambda_1 = \frac{0.7}{2-1} = 0.7, \quad \lambda_2 = \frac{6.5}{10-1} \approx 0.72, \quad F_1(S) = \frac{(1-S)(0.7+S)(1+cS^3)}{2S}.$$

Straightforward computations lead to the formula

$$F_1'(\lambda_1) = \frac{49}{2000}c - \frac{17}{14}.$$

Hence $F_1'(\lambda_1) > 0$ if and only if $c > c_2$ where $c_2 = \frac{17000}{343}$.



$c = c_0 \approx 8.74$ $c = c_1 \approx 10.35$ $c = c_2 = 17000/343 \approx 49.56$

FIG. 3.3. *Graphs of the function $F_1(S)$ for $p_1(S) = 2S/(0.7 + S)$ and $y_1(S) = 1 + cS^3$ in the cases $c = c_0, c_1, c_2$.*

An analysis of the behavior of the function $F_1(S)$ shows (see Fig. 3.3) that there exist two bifurcation values c_0 and c_1, $0 < c_0 < c_1 < c_2$, such that the function $F_1(S)$ is decreasing on $]0, S_0[$ if and only if $0 \leq c \leq c_0$ and the function $F_1(S)$ has two extrema $S_1, S_2 \in]0, \lambda_1[$ satisfying $F_1(S_1) < F_1(\lambda_1) < F_1(S_2)$ if and only if $c \in]c_1, c_2[$.

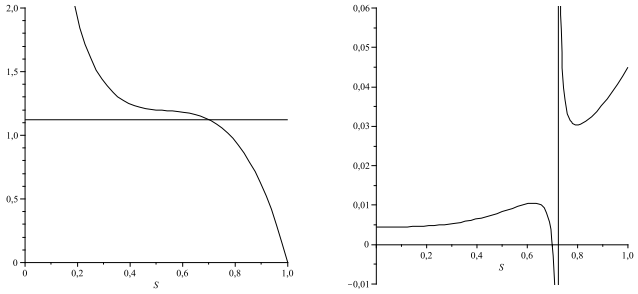


FIG. 3.4. *Graphical verification of hypotheses* 2 *and* 3 *in Theorem 2.2 for (3.6) with $c = 8$ and $m_2 = 10$. On the left, the graph of the function $F_1(S)$. On the right the graph of the function $g_2(S)$.*

COROLLARY 3.2. *The equilibrium E_1^* is locally exponentially unstable if and only if $c > c_2$. If $0 \leq c < c_1$ then the equilibrium E_1^* is globally asymptotically stable .*

Proof. Since $\lambda_1 < \lambda_2$ and $F_1'(\lambda_1) > 0$ if and only if $c > c_2$ the equilibrium E_1^* is locally exponentially stable if and only if $c > c_2$. Hypothesis *3* in Theorem 2.2 is satisfied if and only if $0 \leq c < c_1$ (see Fig. 3.4, left). The function $g_2(S)$ is defined by

$$g_2(S) = w_2(S)\frac{S - \lambda_1}{S - \lambda_2} \text{ where } w_2(S) = \frac{1 + cS^3}{216}.$$

For $c \geq 0$, the function $w_2(S)$ is non-decreasing. By Proposition 2.4, the condition (2.1) with $i = 2$ holds (see Fig. 3.4, right), and the result follows from Theorem 2.2. □

Pilyugin and Waltman showed by numerical simulations that their system exhibits limit cycles in the case where $c = 50$ and $m_2 \geq 9.85$ (see Fig. 4 in [16]). The example

was revisited by Huang, Zhu and Chang [7] who claimed that the limit cycle of the system should remain only on the face $x_2 = 0$ (see [7], Remark 2). We do not agree with this claim. We performed ourselves numerical simulations and actually the limit cycle is contained within the positive cone as shown in Fig. 4 in [16] and not in the face $x_2 = 0$ as claimed in [7].

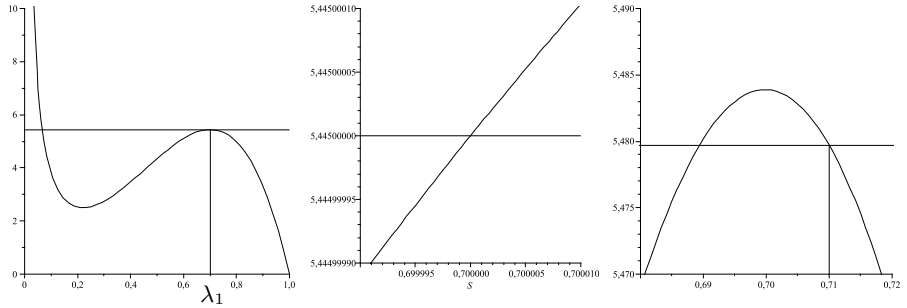


FIG. 3.5. *On the left, the plot of $F_1(S)$ for (3.6) or (3.7) where $c = 50$ and $m_2 = 10$. On the center, the magnification of the neighborhood of $\lambda_1 = 0.7$ shows that $F_1'(\lambda_1) > 0$. On the right, the magnification of the neighborhood of $\lambda_1 = 0.71$ shows that $F_1'(\lambda_1) < 0$.*

Huang, Zhu and Chang [7] made a simple modification by replacing $2S/(0.7+S)$ with $2S/(0.71+S)$ in (3.6) and obtained an example exhibiting competitive exclusion. The model takes the form

$$\begin{aligned} S' &= 1 - S - \tfrac{2S}{0.71+S}\tfrac{x_1}{1+50S^3} - \tfrac{m_2 S}{6.5+S}\tfrac{x_2}{120} \\ x_1' &= [\tfrac{2S}{0.7+S} - 1]x_1 \\ x_2' &= [\tfrac{m_2 S}{6.5+S} - 1]x_2. \end{aligned} \tag{3.7}$$

It is claimed, without proof, in [7] that the equilibrium E_1^* is globally asymptotically stable. Hypothesis *3* in Theorem 2.2 is not satisfied (see Fig. 3.5, left) and we cannot prove the global asymptotic stability of E_1^*. However an explanation of the high sensitivity when 0.7 is replaced by 0.71 is easy to find. Actually the plots of the function $F_1(S)$ in the case of (3.6), where $c = 50$ and (3.7) are very similar (see Fig. 3.5, left), but a magnification of the neighborhood of the value $S = \lambda_1$ shows the differences (see Fig. 3.5, center and right). In (3.6), $F_1'(\lambda_1) > 0$. Hence the equilibrium E_1^* is locally exponentially unstable. In (3.7), $F_1'(\lambda_1) < 0$. Hence the equilibrium E_1^* is locally exponentially stable.

3.5. Further applications. In this section we describe a class of growth functions $p_i(S)$ and yields $y_i(S)$ for which constants α_i satisfying (2.1) exist and hence Theorem 2.2 can be applied. It is convenient to use the notation

$$P_i(S) = \frac{S - \lambda_i}{p_i(S) - D_i} p_i(S) \Longrightarrow p_i(S) = \frac{D_i P_i(S)}{P_i(S) + \lambda_i - S}. \tag{3.8}$$

Remark. We can take any functions $P_i(S)$ that are positive for $0 < S \leq S^0$ and satisfy $P_i(0) = 0$ and use the righthand side of formulas (3.8) to define the functions $p_i(S)$. The function $P_i(S)$ must satisfy the condition $P_i(S) > S - \lambda_i$, so that le denominator in $p_i(S)$ remains positive. If we find a class of yield functions $y_i(S)$ such that the conditions (2.4) hold, where the functions $w_i(S)$, considered in Proposition

2.4, are given by

$$w_i(S) = \frac{y_1(S)}{y_i(S)} \frac{P_i(S)}{P_1(S)}$$

then we can use Proposition 2.4 to obtain the global asymptotic stability of the equilibrium E_1^*.

The Holling type II (Michaelis-Menten or Monod) growth functions

$$p_i(S) = \frac{D_i m_i S}{(m_i - 1)S + \lambda_i} \tag{3.9}$$

correspond to the choice $P_i(S) = m_i$ and $m_i > 1$. The Holling type III (or sigmoidal) growth functions

$$p_i(S) = \frac{D_i m_i S^2}{(a_i + S)(b_i + S)}, \quad \text{with} \quad m_i = \frac{(a_i + \lambda_i)(b_i + \lambda_i)}{\lambda_i^2} \tag{3.10}$$

correspond to the choice $P_i(S) = \frac{(a_i+\lambda_i)(b_i+\lambda_i)S^2}{(a_i+b_i)\lambda_i S+a_i b_i(S+\lambda_i)}$. Here $\mu_i = +\infty$. The prototype for a non-monotone growth function

$$p_i(S) = \frac{D_i m_i S}{(a_i + S)(b_i + S)}, \quad \text{with} \quad m_i = \frac{(a_i + \lambda_i)(b_i + \lambda_i)}{\lambda_i} \tag{3.11}$$

corresponds to the choice $P_i(S) = \frac{(a_i+\lambda_i)(b_i+\lambda_i)}{a_i b_i - \lambda_i S}$. Here $\mu_i = \frac{a_i b_i}{\lambda_i}$. The growth functions (3.9-3.11) were considered by Wolkowicz and Lu [21] who indicated for each combination of them that it is always possible to find appropriate constants α_i^{WL} satisfying the criterion (2.3).

Hereafter we define two new classes of functions, which are not considered in the literature, for which our results apply. A class of monotone growth functions of the form (3.8) is obtained with $P_i(S) = \alpha_i \left[S + \frac{\alpha S}{1+\beta S}\right]$, where $\alpha > 0$, $\beta > 0$ and $\alpha_i \geq 1$. In this case we have

$$p_i(S) = \frac{D_i \alpha_i S(1 + \alpha + \beta S)}{\alpha\alpha_i S + (1 + \beta S)(\alpha_i S - S + \lambda_i)}.$$

For constant yields $y_i(S) = Y_i$ the functions $w_i(S) = \frac{Y_1\alpha_i}{Y_i\alpha_1}$ are constant and hence conditions (2.4) are satisfied. For linear yields (3.3) we have $w_i(S) = \frac{Y_1\alpha_i}{Y_i\alpha_1}\frac{1+c_1 S}{1+c_i S}$. If $c_1 \geq c_i$ then $w_i(S)$ is non-decreasing and hence conditions (2.4) are satisfied. For quadratic yields (3.4) we have $w_i(S) = \frac{Y_1\alpha_i}{Y_i\alpha_1}\frac{1+c_1 S^2}{1+c_i S^2}$. If $c_1 \geq c_i$ then $w_i(S)$ is non-decreasing and hence conditions (2.4) are satisfied.

A class of non-monotone growth functions of the form (3.8) is obtained with $P_i(S) = \alpha_i S^2$ and $\alpha_i > \frac{1}{4\lambda_i}$. In this case we have

$$p_i(S) = \frac{D_i \alpha_i S^2}{\alpha_i S^2 - S + \lambda_i}.$$

For constant yields $y_i(S) = Y_i$ the functions $w_i(S) = \frac{Y_1\alpha_i}{Y_i\alpha_1}$ are constant and hence conditions (2.4) are satisfied. For linear yields (3.3) we have $w_i(S) = \frac{Y_1\alpha_i}{Y_i\alpha_1}\frac{1+c_1 S}{1+c_i S}$. If $c_1 \geq c_i$ then $w_i(S)$ is non-decreasing and hence conditions (2.4) are satisfied. For quadratic yields (3.4) we have $w_i(S) = \frac{Y_1\alpha_i}{Y_i\alpha_1}\frac{1+c_1 S^2}{1+c_i S^2}$. If $c_1 \geq c_i$ then $w_i(S)$ is non-decreasing and hence conditions (2.4) are satisfied.

4. Discussion. In this paper we considered a mathematical model (1.1) of n species of microorganisms in competition in a chemostat for a single resource. The model incorporates both monotone and non-monotone growth functions, distinct removal rates and variable yields. We demonstrated that the CEP holds for a large class of growth functions and yields.

In the case where the yields are constant, it is known [3] that the CEP holds provided that $D_i = D$ for all i, the set $Q = \bigcup_{i\in N}]\lambda_i, \mu_i[$ is connected, and $S^0 \in Q$, where $N = \{i : \lambda_i < S^0\}$. Wolkowicz and Lu [21] conjectured that this result can be extended to the case of different removal rates. Under hypothesis *1* in Theorem 2.2, it is clear that the set Q is connected, and $S^0 \in Q$. The condition $\lambda_1 < \lambda_i$ for $i \neq 1$ in hypothesis *1* can be stated without loss of generality, by labelling the populations such that the index $i = 1$ corresponds to the lowest break-even concentration, but the condition $\lambda_1 < S^0 < \mu_1$ in hypothesis *1* cannot be stated without loss of generality. If $\mu_1 < S^0$, it is not possible to show the CEP by the methods that we used. To the best of our knowledge, in the case of different removal rates and non-monotone growth functions, the CEP has been proved only under the assumption $S^0 < \mu_1$ [9, 21, 22]. However, Rapaport and Harmand [17] considered the case of two populations and proposed conditions on the growth functions such that the CEP holds under the condition $\mu_1 < S^0$. It should be interesting to extend their methods to more general cases. We leave this problem for future investigations.

In the case of constant yields, numerical simulations of model (1.1) have only displayed competitive exclusion. Our results concern also the case of variable yields, for which it is known [1, 7, 16] that more exotic dynamical behaviors, including limit cycles and chaos, are possible. Thus in the case of variable yields, it is of great importance to have criteria ensuring the global convergence to an equilibrium with at most one surviving species. Under certain technical restrictions, we extended the result of Hsu [5] to the case of linear or quadratic yields.

Our proof relies on the construction of non-strict Lyapunov functions, i.e. Lyapunov functions whose derivatives along the trajectories are non-positive. We conjecture that the strictification techniques of Chapter 5 of [11] can be used to construct strict Lyapunov functions, i.e. Lyapunov functions whose derivative along the trajectories are definite negative, which next can be used to establish some robustness properties. This can be the subject of further research.

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REFERENCES

[1] J. Arino, S.S. Pilyugin, G.S.K. Wolkowicz, *Considerations on yield, nutrient uptake, cellular growth, and competition in chemostat models*, Canadian Applied Mathematics Quarterly 11, 2 (2003), 107-142.

[2] R.A. Armstrong, R. McGehee, *Competitive exclusion*, Amer. Natur., 115 (1980), 151-170.

[3] G.J. Butler, G.S.K. Wolkowicz, *A mathematical model of the chemostat with a general class of functions describing nutrient uptake*, SIAM Journal on Applied Mathematics 45 (1985), 138-151.

[4] P. Gajardo, F. Mazenc, H.C. Ramírez, *Competitive exclusion principle in a model of chemostat with delays*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 16 (2009), no. 2, 253–272.

[5] S.B. Hsu, *Limiting behavior for competing species*, SIAM Journal on Applied Mathematics 34 (1978), 760-763.

[6] S.B. Hsu, S.P. Hubbell, P. Waltman, *A mathematical theory for single nutrient competition in continuous culture of micro-organisms*. SIAM Journal on Applied Mathematics 32 (1977), 366-383.

[7] X. Huang, L. Zhu, E.H.C. Chang, *Limit cycles in a chemostat with variable yields and growth rates*, Nonlinear Analysis, Real World Applications, 8, (2007), 165-173.

[8] P. de Leenheer, B. Li, H.L. Smith, *Competition in the chemostat: some remarks*, Can. Appl. Math. Q. 11 (2003), no. 3, 229-248.

[9] B. Li, *Global asymptotic behavior of the chemostat : general response functions and differential removal rates*, SIAM Journal on Applied Mathematics 59 (1998), 411-422.

[10] C. Lobry, F. Mazenc, *Effect on persistence of intra-specific competition in competition models*, Electron. J. Differential Equations 2007, No. 125, 10 pp.

[11] M. Malisoff, F. Mazenc, "Constructions of strict Lyapunov functions", Communications and Control Engineering Series, Springer-Verlag London Ltd, London U.K., 2009.

[12] F. Mazenc, M. Malisoff, J. Harmand *Stabilization in a two-species chemostat with Monod growth functions*, IEEE Trans. Automat. Control 54 (2009), no. 4, 855–861.

[13] F. Mazenc, M. Malisoff, J. Harmand *Further results on stabilization of periodic trajectories for a chemostat with two species*, IEEE Trans. Circuits Syst. I. Regul. Pap. 2008, Special issue on systems biology, 66–74.

[14] F. Mazenc, M. Malisoff, P. De Leenheer, *On the stability of periodic solutions in the perturbed chemostat*, Math. Biosci. Eng. 4 (2007), no. 2, 319–338.

[15] J. Monod, *La technique de culture continue. Théorie et applications*, Ann. Inst. Pasteur, 79 (1950), 390-410

[16] S.S. Pilyugin, P. Waltman, *Multiple limit cycles in the chemostat with variable yields*. Mathematical Biosciences 182, (2003), 151-166.

[17] A. Rapaport, J. Harmand, *Biological control of the chemostat with nonmonotone response and different removal rates*, Math. Biosci. Eng. 5, no. 3 (2008), 539-547.

[18] T. Sari, *Global dynamics of the chemostat with variable yields,* preprint `arXiv 1002.4807v1`

[19] H.L. Smith, P. Waltman, "The Theory of the Chemostat, Dynamics of Microbial Competition". Cambridge University Press, 1995.

[20] G.S.K. Wolkowicz, M.M Ballyk, Z. Lu, *Microbial dynamics in a chemostat : competition, growth, implication of enrichment*, in "Differential Equations and Control Theorey" (Wuhan, 1994), Lecture Notes in Pure and Appl. Math. 176, Dekker, New-York, 1996, pp. 389-406.

[21] G.S.K. Wolkowicz, Z. Lu, *Global dynamics of a mathematical model of competition in the chemostat: general response functions and differential death rates*. SIAM Journal on Applied Mathematics 52 (1992), 222-233.

[22] G.S.K. Wolkowicz, H. Xia, *Global asymptotic behavior of a chemostat model with discrete delays*. SIAM Journal on Applied Mathematics 57 (1997), 1019-1043.